Dynamic Causal Inference & Synthetic Control

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Econometrics/Reading Session

Potential Outcome Framework: Time series

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Part I

Potential Outcome Framework: Time series

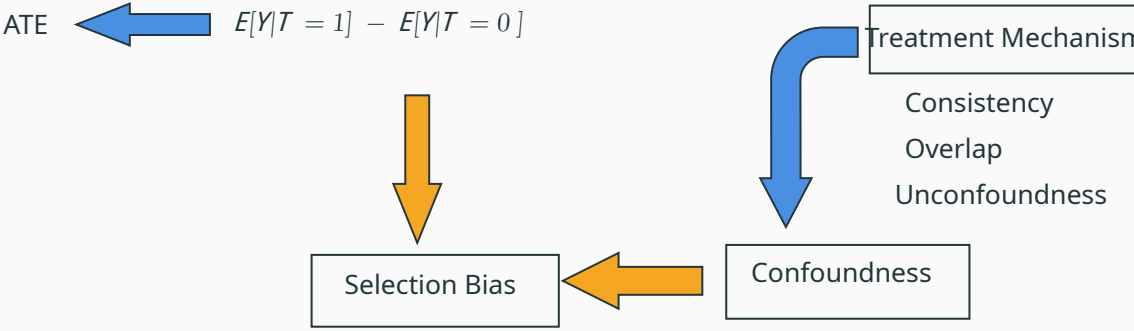
Recap

Quick Recap

- The core of inferring Causality lies in uncovering ”Counterfactuals”.

$$Y(T = 1) - Y(T = 0)$$

- **Key:** The central idea behind quantifying causal effects lies in the Treatment Mechanism. Holland, 1986



What about Dynamic Systems

Dynamic Causal Effects

- In dynamic settings, quantifying causality is more complex.
  - Both treatment and outcome evolve over time.
  - Dynamic feedback between treatment and outcome.
- Traditionally, causal relationships in macro literature relied on structural models.
  - Restrictions based on economic theory are imposed to identify causal effects.
  - Much of the literature relies on parametric linear models.
  - Widely used framework includes SVARs and Local Projections
- The strong functional form assumptions may implicitly impose substantial restrictions on the underlying dynamic causal effects.

Setup

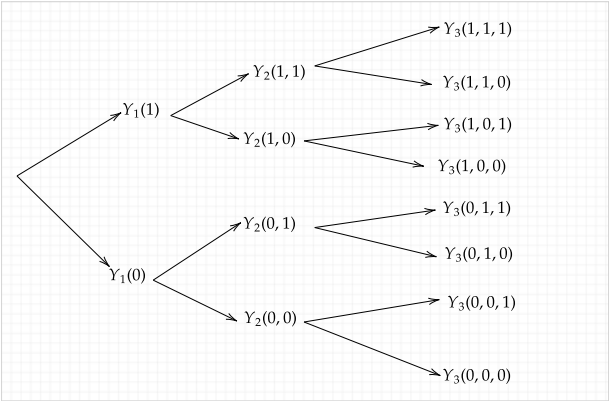
Potential Outcome Time series Framework

- Consider a single unit.
- At each time $t = 1, \cdots T$, the unit is exposed to a scalar treatment $W_t \in \mathcal{W} = \{0, 1\}$.
- Define the stochastic process $\{W_t\}_{t=1}^{T}$ as the **Assignment Process**, where each $W_t \in \mathcal{W}$.
- Potential Outcome Process: For any deterministic $\omega_{1:T} \in \mathcal{W}^T$, the stochastic process

$$Y_{1:T}(\omega_{1:T}) := (Y_1(\omega_{1:T}), Y_2(\omega_{1:T}), \ldots, Y_T(\omega_{1:T}))^\top$$

- Each time t potential outcome $Y_t(\omega_{1:T}) : \mathcal{W}^T \to \mathbb{R}$
- This definition shows that the potential outcome can depend on future treatments.
- Think of time t potential outcome process as the cross-sectional framework in which the outcome of unit i depends on the treatment of everyone.

Illustration



Assumption (Non-Anticipating {No Interference})

For any deterministic $\{\omega_t\}_{t\geq 1}, \{\omega'_t\}_{t\geq 1}$ *with* $\omega_t, \omega'_t \in \mathcal{W}$*;*

$$Y_t\left(\omega_{1:t}, \{\omega_s\}_{s\geq t+1}\right) = Y_t\left(\omega_{1:t}, \{\omega'_s\}_{s\geq t+1}\right) \quad \forall t \geq 1 \quad a.s$$

Assumption (Output {Consistency})

The output process $\{W_t, Y_t\}_{t\geq 1} = \{W_t, Y_t(W_{1:t})\}_{t\geq 1}$*. The* $\{Y_t\}_{t\geq 1}$ *is called the outcome process.*
In short, $Y_{1:T} = (Y_1(W_1), Y_2(W_{1:2}), \cdots Y_T(W_{1:T}))$ *is the random process whose realization is observed.*

Assumption (Sequential Probabilistic assignment{Overlap})

$0 < P(W_t = \omega \mid \mathcal{F}_{t-1}) < 1$ *with probability one for all* $w \in \mathcal{W}$*.*
$\mathcal{F}_t$ *is determined by* $\left\{W_t, \left\{Y_t(\omega_{1:t}), \omega_{1:t} \in \mathcal{W}^t\right\}\right\}_{t\geq 1}$*.*

Direct Potential Outcome System

Definition

Any $\left\{W_t, \left\{Y_t\left(\omega_{1:t}\right) : \omega_{1:t} \in \mathcal{W}^t\right\}\right\}_{t\geq 1}$ satisfying Assumptions 1-3 is a direct potential outcome system.

- **Non-Anticipating on Assignment Process:** Assume that W_t does not explicitly influence future assignment W_s , for $s > t$.
- W_t should directly affect potential outcome process Y_t but not indirectly through future assignment.
- Dynamic causal effect at time t: $Y_t(\omega_{1:t}) - Y_t(\omega'_{1:t})$
- Notation: Define potential outcome for time $t + h$ as

$$Y_{t+h}(\omega) := Y_{t+h}\left(W_{1:t-1}, \omega, W_{t+1:t+h}\right)$$

- For $t \geq 1, h \geq 0$, and any fixed $\omega, \omega' \in \mathcal{W}$, the time- t, h-period ahead:
  - Impulse causal effect : $Y_{t+h}(\omega) - Y_{t+h}\left(\omega'\right)$
  - Average Treatment Effect: $\mathbb{E}\left[Y_{t+h}(\omega) - Y_{t+h}\left(\omega'\right)\right]$

Application in Macroeconomics

Impulse Response Function

- Quantifying the effect of Monetary Policy on Economy.
- In these literature, the quantity of interest is Impulse response function.

$$IRF = \mathbb{E}\left[Y_{t+h} \mid W_t = \omega\right] - \mathbb{E}\left[Y_{t+h} \mid W_t = \omega'\right]$$

- Traditionally, the IRF has been estimated by modeling the entire dynamic system (VAR's).
- Recently, Local Projections estimate IRF using by running simple OLS.
- Two Questions poised by Rambachan and Shephard, 2021
  - When is IRF causal?
  - How can we estimate IRF non-parametrically?
- Spoiler Alert: The IRF is causal only under randomization of the Treatment.

Estimation

- The IRF has a causal interpretation if the shock or intervention is exogenous.
- In our framework, this holds when

$$W_t \perp \left(W_{1:t-1}, W_{t+1:t+h}, \left\{Y_{t+h}\left(w_{1:t+h}\right) : w_{1:t+h} \in \mathcal{W}^{t+h}\right\}\right)$$

- This is equivalent to the Ignorability assumption or CIA.
- Based of this assumption:
  - Angrist et al., 2018 employed a semiparametric estimates of monetary policy effects using Inverse probability weighting estimators.
  - Recently, Ballinari and Wehrli, 2024 employs Double debiased machine learning to estimate the IRF.

Remarks

- The Direct Potential Outcome system framework places no functional form restrictions.
- No common time series assumptions such as invertibility.
- The VAR and LPs methodologies are special case of this framework.

Part II

The Synthetic Control Methods

Introduction

Motivation

- Quantify the effects of policy intervention that takes place at an Aggregate level (States, country, Firms).
- Study the effect of terrorist conflicts in Basque country on the Basque economy. Abadie and Gardeazabal, 2003
- The intervention affects all units. (Overlap assumption is violated.)
- Comparative case studies: Compare the evolution of an aggregate outcome for the treated unit and control unit. Card, 1993
- The main limitation:
  - Degree of ambiguity in the choice of comparison group.
  - The inferential techniques of these comparison case studies only quantifies sampling uncertainty.
  - No inference on the uncertainty about the control group being a good counterfactual for the treated unit.
- Do not Capture heterogeneous effect across different subpopulation. Gunsilius, 2023

Synthetic Control

- Synthetic control (SC) proposed **abadie2023**; Abadie et al. (2010) reduces the ambiguity and subjectivity by quantifying how affinity.
- In practice, it is difficult to find a single unit that best matches the most relevant characteristics of the treated unit.
- Uses a combination of control units (donor pool) to approximate the counterfactual for treated units.
- Captures the relative contribution of each control unit to the counterfactual.

Setup

Famework

- Suppose that we observe $J + 1$ units in periods $1, 2, \cdots T$.
- Unit one is exposed to the intervention of interest (that is,"treated") during periods $T_0 + 1, \cdots T$.
- The remaining J units are an untreated reservoir of potential controls (a "donor pool").
- Let Y_{it}^I be the outcome that would be observed for unit i at time t if unit i is exposed to the intervention in periods $T_0 + 1$ to T.
- Let Y_{it}^N be the outcome that would be observed for unit i at time t in the absence of the intervention.
- We aim to estimate the effect of the intervention on the treated unit,

$$\tau_{1t} = Y_{1t}^I - Y_{1t}^N = Y_{1t} - Y_{1t}^N$$

for $t > T_0$, and Y_{1t} is the outcome for unit one at time t.

Assumption

- *No Anticipation Assumption* *i.e.* $Y_{it}^{I} = Y_{it}^{N}$ *for* $t \in \{1, \cdots T_0\}$
- *Sequential Treatment:* *The treatment does not turn off i.e. it continues to affect the treated unit in all* $T_0 + 1, \cdots, T$
- *No Interference Assumption*

Implementation

Implementation

- A SC is a weighted average of units from a donor pool.
- Let $\boldsymbol{W} = (w_2, \ldots, w_{J+1})'$ s.t. $w_j \geq 0$ for $j = 2, \ldots, J+1$ and $w_2 + \cdots + w_{J+1} = 1$.
- The goal is to obtain $\left(w_2^*, \ldots, w_{J+1}^*\right)$ such that:

$$\begin{array}{ll} \sum_{j=2}^{J+1} w_j^* Y_{j1} = Y_{11}, & \sum_{j=2}^{J+1} w_j^* Y_{j2} = Y_{12}, \\ \sum_{j=2}^{J+1} w_j^* Y_{jT_0} = Y_{1T_0}, & \text{and} \quad \sum_{j=2}^{J+1} w_j^* \mathbf{Z}_j = \mathbf{Z}_1 \end{array}$$

- Let $\mathbf{X}_1 = \left(\mathbf{Z}_1', \bar{Y}_1^{\mathbf{K}_1}, \ldots, \bar{Y}_1^{\mathbf{K}_M}\right)'$ be a $(k \times 1)$ vector of pre-intervention characteristics for the treated unit. Similarly, let $\mathbf{X}_0$ is $\left(\mathbf{Z}_j', \bar{Y}_j^{\mathbf{K}_1}, \ldots, \bar{Y}_j^{\mathbf{K}_M}\right)'$ be a $(k \times J)$ matrix which contains the same variables for the unaffected units.
- The vector $\boldsymbol{W}^* = \left(w_2^*, \ldots, w_{J+1}^*\right)'$ is chosen to minimize $\|\boldsymbol{X}_1 - \boldsymbol{X}_0 \boldsymbol{W}\|$, subject to our weight constraints.
- $\|\mathbf{X}_1 - \mathbf{X}_0 \mathbf{W}\|_{\mathbf{V}} = \sqrt{(\mathbf{X}_1 - \mathbf{X}_0 \mathbf{W})' \mathbf{V} (\mathbf{X}_1 - \mathbf{X}_0 \mathbf{W})}$

Implementation (cont.)

- V represents a symmetric positive semi-definite matrix chosen by the researcher or by data-driven methods (out-of-sample validation).
- For a post-intervention period t (with $t > T_0$) the SC estimator is:

$$\widehat{\tau}_{1t} = Y_{1t} - \sum_{j=2}^{J+1} w_j^* Y_{jt}$$

Properties of SCM

Motivating Model

- Recall that $\tau_{1t} = Y_{1t} - Y_{1t}^N$
- The SC estimator is a finite sample estimator i.e. so asymptotic properties may not apply in this context.
- To evaluate the validity of the SC estimator, we compute the $Bias(\tau_{1t})$.
- suppose we model the counterfactual with a factor model :

$$Y_{it}^N = \theta_t Z_{it} + \lambda_t \mu_i + \varepsilon_{it}$$

where $\boldsymbol{Z}_i$ are observed features, $\boldsymbol{\lambda}_t$ are unobserved factors, and ε_{it} is a unit-level transitory shock, modeled as random noise.

- This generalizes the DiD method as it allows for unobserved confounders to vary over time.
- As pointed out earlier, SCM seeks for $\{w\}_{j=2}^{J+1}$ s.t

$$\sum_{j=2}^{J+1} w_j^* Y_{jt} = Y_{1t} \quad \forall t = 1, .., T_0$$

- If this match holds, Bias = 0.

- This match holds exactly only if $(Y_{11}, \ldots, Y_{1T_0}, \mathbf{Z}_1')$ belongs to the convex hull of $\{(Y_{21}, \ldots, Y_{2T_0}, \mathbf{Z}_2'), \ldots, (Y_{J+11}, \ldots, Y_{J+1T_0}, \mathbf{Z}_{J+1}')\}$.

Proposition

Suppose that $E\left|\varepsilon_{jt}\right|^p < \infty$ *for some* $p > 2$. *Let* $\sigma_{jt}^2 = E\left|\varepsilon_{jt}\right|^2$, $\sigma_j^2 = (1/T_0)\sum_{t=1}^{T_0}\sigma_{jt}^2$, $\bar{\sigma}^2 = \max_{j=2,\ldots J+1}\sigma_j^2$, *and* $\bar{\sigma} = \sqrt{\bar{\sigma}^2}$. *Similarly, let* $m_{p,jt} = E\left|\varepsilon_{jt}\right|^p$, $m_{p,j} = (1/T_0)\sum_{t=1}^{T_0} m_{p,jt}$, *and* $\bar{m}_p = \max_{j=2,\ldots J+1} m_{p,j}$. *We obtain that, for* $t > T_0$,

$$|E[\widehat{\tau}_{1t} - \tau_{1t}]| \leq C(p)^{1/p}\left(\frac{\bar{\lambda}^2 F}{\underline{\xi}}\right) J^{1/p} \max\left\{\frac{\bar{m}_p^{1/p}}{T_0^{1-1/p}}, \frac{\bar{\sigma}}{T_0^{1/2}}\right\}$$

- The bias of the estimator can be bounded by a function that goes to zero as the number of pretreatment periods increases.

Important Remarks of SC

- Proposition 1 demonstrates that the credibility of the synthetic control depends extent to which it can fit the trajectory of Y_{1t} for an extended pre-intervention period.
- There is no ex-ante guarantee that SC will be a good fit.
- This happens when $(Y_{11}, \ldots, Y_{1T_0}, \mathbf{Z}_1')$ falls far from the convex hull of $\{(Y_{21}, \ldots, Y_{2T_0}, \mathbf{Z}_2'), \ldots, (Y_{J+11}, \ldots, Y_{J+1T_0}, \mathbf{Z}_{J+1}')\}$.
- If the fit is poor, Abadie et al., 2010 recommends against the use of Synthetic control .
- Settings with small T_0, large J, and large noise create substantial risk of over-fitting.
- We can reduce interpolation bias by restricting the donor pool to regions with similar characteristics to the treated.

Empirical Implementation

Tobacco Control Program

- In 1988, California passed proposition 99, anti-tobacco law.
  - Introduced a 25-cent per pack state excise tax.
  - Banned vending machines and single cigarette sales.
  - Revenue-funded health and anti-tobacco programs.
- We investigate the impact of Proposition 99 on smoking prevalence using synthetic control methodology.

Data and Sample

- Annual state-level data from 1970 - 2000.
- Proposition was implemented in January 1989, $\implies T_0 = 19$ years .
- Create a Synthetic California counterfactual, using the remaining 49 states as donor Pool during the preintervention periods.
- Donor Pool is restricted to 38 states, as some states did implement anti-tobacco policies.
- The Outcome of interest is Annual per capita cigarette sales.
- Included Predictors such as ”Average Retail Price of Cigarette”, ”Per-capita state income”, and ”Per-capita beer consumption”.
- Choose $\bar{Y}_i^K = (CigSales_{1975}, CigSales_{1980}, CigSales_{1988})$

Traditional Comparison group

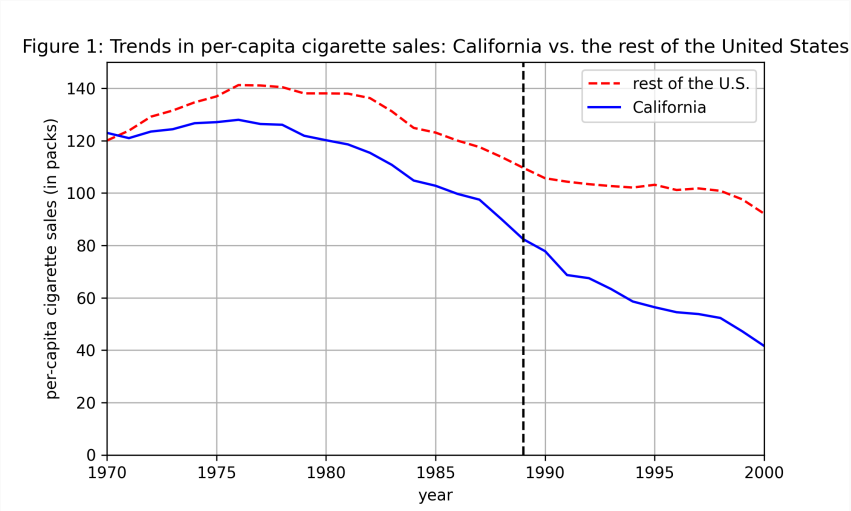


Figure 2: Average of all units in donor pool as a Comparison group

Synthetic Control

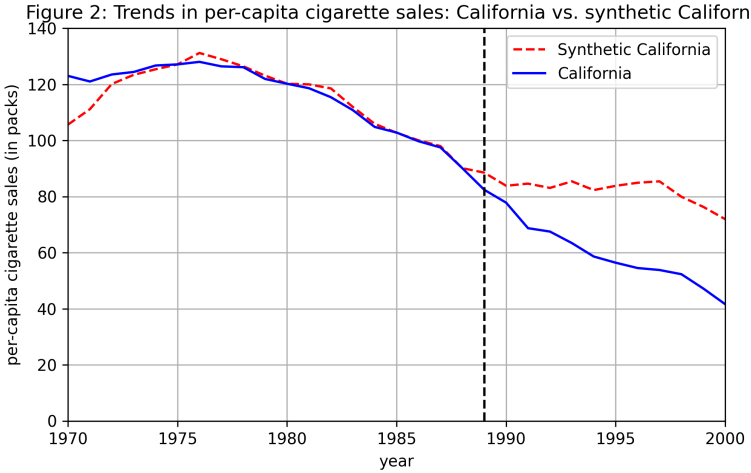


Figure 3: Trends in cigarette sales of the Synthetic california and Carlifonia

Treatment effect

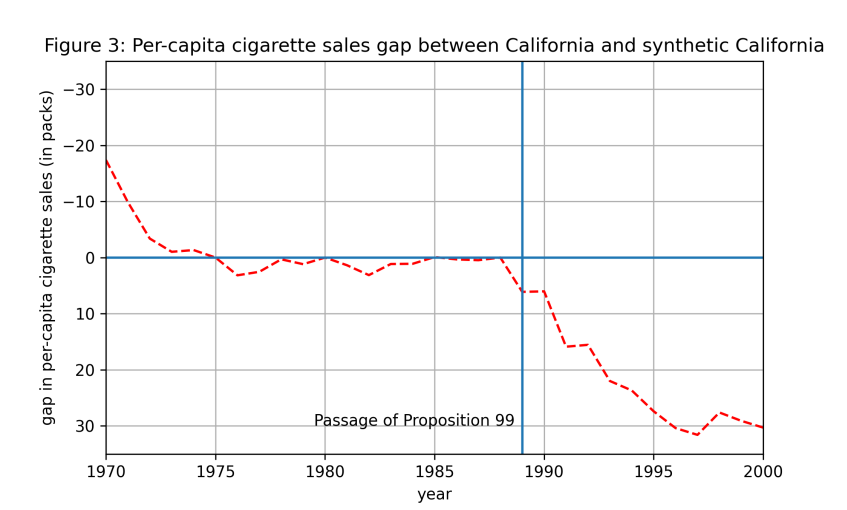


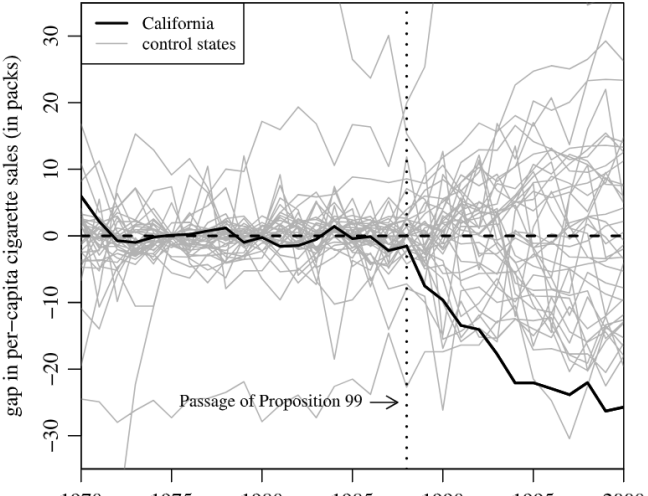
Figure 4: Impact of Proposition 99 on cigarette sales

Inference

Inference

- To evaluate the significance of our estimates, we pose the question of whether our results could be driven entirely by chance.
- Large sample inferential techniques is not well suited for this methodology.
- Abadie et al. (2010) propose a mode of inference for the synthetic control framework that is based on permutation methods.
- A permutation distribution can be obtained by iteratively reassigning the treatment to the units in the donor pool and estimating placebo effects in each iteration.
- The effect of the treatment on the unit affected by the intervention is deemed to be significant when its magnitude is extreme relative to the permutation distribution.

Placebo Test



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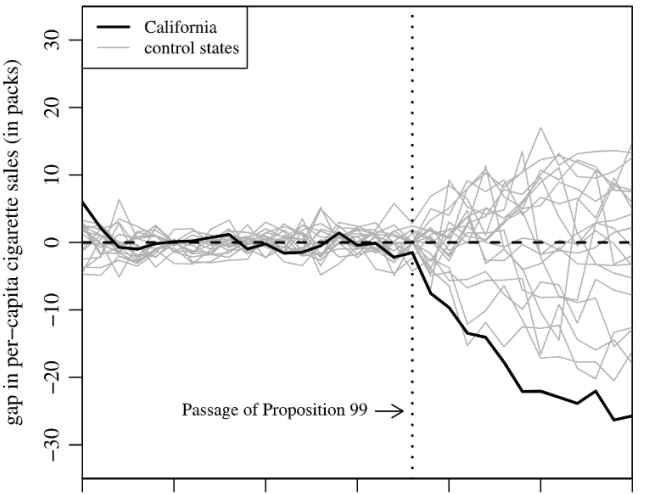

Conclusion

Comment

- Can the Synthetic control methods be extended to identifying IRF?
- Recall that the dynamic causal effect requires a comparison between the observed treatment path and the comparison treatment path.
- Recent literature by Gonçalves et al., 2024 provides a new definition of IRF which takes an average over all possible comparison of the treatment paths.
- Perhaps a Synthetic control method can be extended to provide a synthetic version of the treatment path as a comparison to the observed path??

Thanks!

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References

- Abadie, A., Diamond, A., & Hainmueller, J. (2010). **Synthetic control methods for comparative case studies: Estimating the effect of california's tobacco control program.** *Journal of the American statistical Association*, *105*(490), 493–505.
- Abadie, A., & Gardeazabal, J. (2003). **The economic costs of conflict: A case study of the basque country.** *American economic review*, *93*(1), 113–132.
- Angrist, J. D., Jordà, Ò., & Kuersteiner, G. M. (2018). **Semiparametric estimates of monetary policy effects: String theory revisited.** *Journal of Business & Economic Statistics*, *36*(3), 371–387.
- Ballinari, D., & Wehrli, A. (2024). **Semiparametric inference for impulse response functions using double/debiased machine learning.** *arXiv preprint arXiv:2411.10009*.

- Card, D. (1993). **Minimum wages and employment: A case study of the fast food industry in new jersey and pennsylvania.**
- Gonçalves, S., Herrera, A. M., Kilian, L., & Pesavento, E. (2024). **State-dependent local projections.** *Journal of Econometrics*, 105702.
- Gunsilius, F. F. (2023). **Distributional synthetic controls.** *Econometrica*, *91*(3), 1105–1117.
- Holland, P. W. (1986). **Statistics and causal inference.** *Journal of the American statistical Association*, *81*(396), 945–960.
- Rambachan, A., & Shephard, N. (2021). **When do common time series estimands have nonparametric causal meaning.** *Manuscript, Harvard University*.