The Journey of Counterfactuals

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Econometrics/Reading Session

Causality and Counterfactuals

Causality and Counterfactuals

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- The core of inferring Causality lies in uncovering ”Counterfactuals”.
- Counterfactuals represent a hypothetical scenario of what an outcome would have been in a different state of the world.Heckman and Pinto, 2024
- The treatment status defines the hypothetical state.
- Policymakers: Examining the impact of some treatment on an Outcome.
- What are the long-run impacts of the OLPC program on Test scores? Cueto et al., 2024
- This causal quantification can be inferred from Counterfactual comparisons.

$$Y(T = 1) - Y(T = 0)$$

- Key component: The only difference between these hypothetical states is the *treatment assignment*.

Illustration

Potential Outcomes Framework

- Consider a population of N units
- Let $T_i \in \{0, 1\}$ denote unit i's treatment.
- $Y_i(1)$ represents the potential outcome of unit i at the Treated state.
- $Y_i(0)$ represents the potential outcome of unit i at the untreated state.
- Unit i causal effect is $ICE = Y_i(1) - Y_i(0)$.
- Implicit in this setup is the SUTVA assumption. more
- ICE is unobserved.
- Infer ICE from distribution treatment effect i.e

$$ATE = \mathbb{E}(Y_i(1) - Y_i(0))$$

The Problem of Missing Data

- For each unit, only one potential outcome is observed.

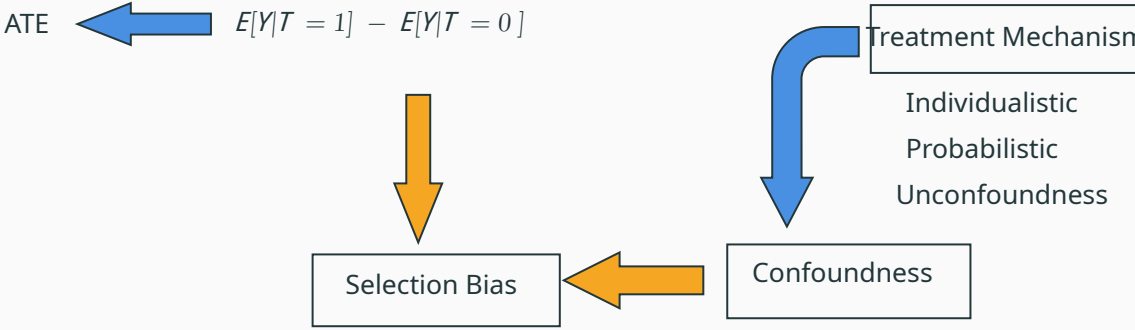
$$Y_i^{obs} = T_i Y_i(1) + (1 - T_i) Y_i(0)$$

- Implication of missing data is the Problem of Selection Bias

$$\begin{aligned}\mathbb{E}\left(Y_i \mid T = 1\right) - \mathbb{E}(Y_i \mid T = 0) = \mathbb{E}\left(Y_i(1) \mid T = 1\right) - \mathbb{E}\left(Y_i(0) \mid T = 1\right) \\ + \mathbb{E}\left(Y_i(0) \mid T = 1\right) - \mathbb{E}\left(Y_i(0) \mid T = 0\right).\end{aligned}$$

- Selection Bias occurs because there exists a systematic difference between the treated group and the control.
- Selection Bias is a problem of Confoundness. Mahdavi and Ehsani, 2023
- Confoundness is dealt with by restrictions on Treatment Mechanism.

Big Picture



- **Key:** The central idea behind quantifying causal effects lies in the Treatment Mechanism. Holland, 1986
- **Restrictions:** Restrictions on the treatment mechanism, either by design or assumption, are required to deal with the problem of confoundness.
- Heckman and Pinto, 2024 and McDonald, 2002

Potential Outcome Vs Do Calculus Framework

- According to Heckman and Pinto, 2024, The Potential Outcome framework is agnostic about the relationship between Treatment and the Potential Outcome.
- The Framework assumes $Y(1), Y(0)$ as primitive constructs assumed to exist without a model that describes this relationship.
- Assumptions and Restrictions must be made to describe the relationship between Treatment and Outcome. ”No Causality without manipulation”Holland, 1986.
- The do calculus framework developed by Pearl, 2009 describes the relationship between Treatment and Potential Outcome using structural equation models.
- How is this framework different from Structural VAR models in time series?

Treatment Mechanism

Definition (Assignment Mechanism)

The Assignment mechanism is a row-exchangeable function $\Pr(T \mid X, Y(0), Y(1))$ taking values $\in [0, 1]$, satisfying

$$\sum_{T \in \{0,1\}^N} \Pr(T \mid \mathbb{X}, \mathbb{Y}(0), \mathbb{Y}(1)) = 1 \quad \forall \quad \mathbb{X}, \mathbb{Y}(0), \mathbb{Y}(1).$$

$$p_i(T \mid \mathbb{X}, \mathbb{Y}(0), \mathbb{Y}(1)) = \sum_{\mathbb{T}: T_i = 1} \mathrm{P}(T \mid \mathbb{X}, \mathbb{Y}(0), \mathbb{Y}(1))$$

- Individualistic: $P_i(X, Y(0), Y(1)) = q\,(X_i, Y_i(0), Y_i(1))$
- Probabilistic: $0 < \quad P_i(x, \mathbb{Y}(0), Y(1)) < 1$
- Unconfoundness: $\Pr(\mathbb{T} \mid \mathbb{X}, \mathbb{Y}(0), \mathbb{Y}(1)) = \Pr(\mathbb{T} \mid \mathbb{X}, \mathbb{Y}'(0), \mathbb{Y}'(1))$

Sufficient Condition

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Assumption

Given the DGP (X, T, Y), the causal estimand $\tau_{ATE} = E[Y(1)] - E[Y(0)]$ can be inferred from $E[E_X[Y|A = 1, X]] - E[E_X[Y|A = 0, X]]$ under the following condition: Pan, 2024

1. ***Positivity:*** *The probability of being assigned to the treatment and control groups, conditioned on the covariates, is a positive number between 0 and 1:*

$$P(T = 1|X) \in (0, 1), \quad P(T = 0|X) \in (0, 1).$$

2. ***Consistency:*** *The potential outcome under the treatment received is the same as the observed outcome:*

$$Y = Y(T).$$

3. ***Unconfoundedness:*** *Conditional on a set of observed covariates X, the potential outcomes $Y(1)$ and $Y(0)$ are independent of the treatment assignment T:*

$$\{Y(1), Y(0)\} \perp\!\!\!\perp T \mid X.$$

RCT Methodology

- Treatment assignment is controlled by the Researcher and done by randomization.
- The treatment mechanism is random, i.e., all treatment assignments have an equal probability of occurring.
- Hence the treatment by design is independent of the Potential Outcomes.
- Intuitively, RCT eliminates systematic differences between groups by randomizing the treatment.
- $ATE = \mathbb{E}(\mathbb{E}(Y_i|T = 1) - \mathbb{E}(Y_i|T = 0))$

Observational Data

Pseudo Randomized Methods

- Pseudo-randomized methods emulate randomization by imposing assumption 1 on the treatment mechanism.
- These methods ensure that treated and control units are comparable, reducing or eliminating selection bias.
- These methods include:
  - Matching
  - Inverse Probability weighting
  - Propensity Score Methods.
- These methods rely on the overlap assumption which fails in settings with Aggregate Treatment.

Violation of Positivity Assumption

- Suppose we want to quantify the effects of rules in sports gambling on suicide risks for the residents of a state.
- The Positivity assumption is violated.
- causal estimand is unidentifiable.
- A class of methodologies with further assumptions are required:
  - Regression Discontinuity Design.
  - Difference-in-Difference
  - Synthetic Control.

- Treatment assignment is deterministic i.e. $T = 1\{X \geq c\}$
- Globally, there is no overlap between treated and untreated groups.
- Although, we cannot recover global ATE, the continuity assumption enables us to infer local causal effect at the cutoff point.

Assumption

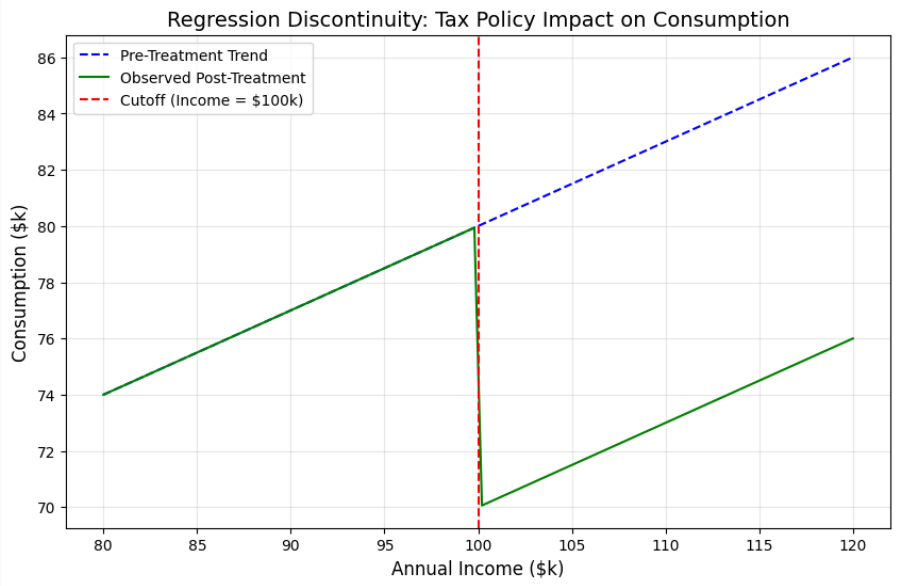
The expected potential outcomes $E[Y(0) \mid X = x]$ and $E[Y(1) \mid X = x]$ are continuous at the threshold c :

$$\lim_{x \to c^-} E[Y(0) \mid X = x] = \lim_{x \to c^+} E[Y(0) \mid X = x]$$

$$\lim_{x \to c^-} E[Y(1) \mid X = x] = \lim_{x \to c^+} E[Y(1) \mid X = x]$$

$$ATEC = \lim_{x \to c^-} E\left(Y_i \mid X_i = x\right) - lim_{x \to c^+} E\left(Y_i \mid X_i = x\right),$$

Illustration



Difference-in-Difference

- Consider the 2×2 canonical model with two time periods $t = 1, 2$.
- Two Groups: The treated group ($T_i = 1$) and the control group($T_i = 0$).
- Treatment occurs at $t = 2$.
- **Parameter of interest:**

$$ATT \equiv \mathbb{E}\left[Y_{i,2}(1) \mid T_i = 1\right] - \mathbb{E}\left[Y_{i,2}(0) \mid T_i = 1\right]$$

- **Parallel Trends Assumption:**

$$\mathbb{E}\left[Y_2(0) - Y_1(0) \mid T = 1\right] = \mathbb{E}\left[Y_2(0) - Y_1(0) \mid T = 0\right]$$

- **No Anticipation Assumption:** $Y_{i,1}(0) = Y_{i,1}(1)$ for all i with $T_i = 1$
- Using the PT and NA, we infer the counterfactual:

$$\mathbb{E}\left[Y_{i,2}(0) \mid T_i = 1\right] = \mathbb{E}\left[Y_{i,1}(1) \mid T_i = 1\right] + \mathbb{E}\left[Y_{i,2}(0) - Y_{i,1}(0) \mid T_i = 0\right]$$

- $ATT = \mathbb{E}\left[Y_{i,2} - Y_{i,1} \mid T_i = 1\right] - \mathbb{E}\left[Y_{i,2} - Y_{i,1} \mid T_i = 0\right]$

As a regression

- As Economists we love regressions!
- We can use a simple regression (TWFE) to estimate θ, the ATT:

$$Y_{i,t} = \alpha_i + \phi_t + (1[t = 2] \cdot D_i)\,\beta + \epsilon_{i,t}$$

- We can verify that $\hat{\beta} \equiv ATT$
- We can leverage its regression representation to conduct asymptotically valid inference.

In empirical settings

- Although the canonical setup is very easy to understand, the empirical applications in the real world present some differences.
  - More than 2 time periods.
  - Variation in treatment time.
  - No absorption treatment (e.g., turn on/turn off).
  - Parallel trends may not hold with different functional forms.
- Recent literature has shown how these variations present problems in traditional methods.
- For more information: Callaway and Sant'Anna, 2021; de Chaisemartin and D'Haultfœuille, 2020; Sun and Abraham, 2021

Preview to Synthetic Control

Motivation

- The Quasi-randomized methods is limited in settings where treatment affects the majority of the population or aggregate.
- In such cases Positivity assumption is violated and the counterfactuals cannot be inferred.
- Synthetic Control Construct a counterfactual for the treated aggregate using a weighted combination of untreated units.
- Unlike matching and PS methods, SC does not require explicit assumptions about the treatment.
- This allows researchers to study the causal effects of large-scale interventions even when traditional methods fail.

Synthetic Control

- RDD estimates the local average treatment effect (LATE) at the cutoff point, which may not generalize to the entire population.
- RDD requires sufficient observations close to the threshold for credible estimates, which is often unavailable.
- In DiD setups for aggregate interventions, the choice of the control group can be subjective.
- The SC methods reduce the subjectivity of control units by using data-driven and transparent process.
- SC methods allows for unobserved factors that varies over time.

Preview into Synthetic Control

- Synthetic control (SC) methods were originally proposed in Abadie and Gardeazabal (2003) and Abadie et al. (2010a) with the goal to estimate the effect of aggregate interventions.
- Many interventions of interest naturally happen at an aggregate level affecting a small number of large units (such as cities, regions, or countries).
- When the units of analysis are a few aggregate entities, a combination of comparison units (a ”synthetic control”) often does a better job reproducing the characteristics of a treated unit than any single comparison unit alone.
- A SC is selected as the weighted average of all potential comparison units that best resemble the characteristics of the treated unit(s).

Famework

- Suppose that we observe $J + 1$ units in periods $1, 2, \cdots T$.
- Unit one is exposed to the intervention of interest (that is,"treated") during periods $T0 + 1, \cdots T$.
- The remaining J units are an untreated reservoir of potential controls (a "donor pool").
- Let Y_{it}^{I} be the outcome that would be observed for unit i at time t if unit i is exposed to the intervention in periods $T_0 + 1$ to T.
- Let Y_{it}^{N} be the outcome that would be observed for unit i at time t in the absence of the intervention.
- Assume No Anticipation Assumption i.e. $Y_{it}^{I} = Y_{it}^{N}$ for $t \in \{1, \cdots T_0\}$
- We aim to estimate the effect of the intervention on the treated unit,

$$\tau_{1t} = Y_{1t}^{I} - Y_{1t}^{N} = Y_{1t} - Y_{1t}^{N}$$

for $t > T_0$, and Y_{1t} is the outcome for unit one at time t.

Framework (cont.)

- As we said before, SC is a weighted average of units from a donor pool.
- Let $\boldsymbol{W} = (w_2, \ldots, w_{J+1})'$ with $w_j \geq 0$ for $j = 2, \ldots, J+1$ and $w_2 + \cdots + w_{J+1} = 1$. Each value of $\boldsymbol{W}$ represents a potential synthetic control.
- Let $\boldsymbol{X}_1$ be a $(k \times 1)$ vector of pre-intervention characteristics for the treated unit. Similarly, let $\boldsymbol{X}_0$ be a $(k \times J)$ matrix which contains the same variables for the unaffected units.
- The vector $\boldsymbol{W}^* = \left(w_2^*, \ldots, w_{J+1}^*\right)'$ is chosen to minimize $\|\boldsymbol{X}_1 - \boldsymbol{X}_0 \boldsymbol{W}\|$, subject to our weight constraints.
- Let Y_{jt} be the value of the outcome for unit j at time t. For a post-intervention period t (with $t \geq T_0$) the SC estimator is:

$$\widehat{\tau}_{1t} = Y_{1t} - \sum_{j=2}^{J+1} w_j^* Y_{jt}$$

Framework (cont.)

- Typically,

$$\|\mathbf{X}_1 - \mathbf{X}_0\mathbf{W}\|_{\mathbf{V}} = \sqrt{(\mathbf{X}_1 - \mathbf{X}_0\mathbf{W})' \mathbf{V} (\mathbf{X}_1 - \mathbf{X}_0\mathbf{W})}$$

- *V* represents a symmetric positive semi-definite matrix chosen by the researcher or by data-driven methods (out-of-sample validation).

Expectation for future Presentations

Expectation

- Review the SC methodology, estimation, and inference procedure.Abadie et al., 2010b
- Review key challenges and issues with traditional SCM in particular, address the issue of correlated errors in the SCM framework.Medeiros, 2024 and Gonçalves and Ng, 2024
- Review the inference procedure in a more general setting, sensitivity analysis and confidence sets in SCM.Firpo and Possebom, 2018
- Synthetic Control from Methodological Viewpoint. Abadie, 2021
- Synthetic control in high dimensional settings. Factor-Adjusted Regularized Method for Treatment Evaluation. Fan et al. (2022)
- Counterfactual Analysis and Inference with Nonstationary Data.Masini and Medeiros (2022)

Conclusion

- Restriction of the Treatment mechanism is crucial in identifying causal effects.
- Causal Inference Methodologies can be classified based on Restrictions.
- The no overlap assumption is violated in many settings leading us to the study of Synthetic control methods.

Assumption

The potential outcomes for any unit do not vary with the treatments assigned to other units, and, for each unit, there are no different forms or versions of each treatment level, which lead to different potential outcomes. back to PO

- No Interference: $Y_i(\mathbb{T}) = Y_i(T_i)$, for $\mathbb{T} = (T_1 . \cdots T_N)$
- Consistency: For each unit i, there are no different forms or versions of each treatment level, which lead to different potential outcomes.

Thanks!

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